Braids Groups and Representation Theory

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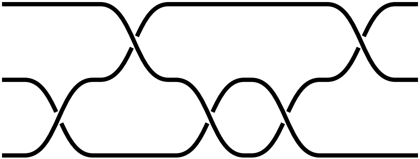
Group actions on Vector Spaces and Braids

Permuation action of S_n on $V = \mathbb{C}^n$ gives rise to a braid group

$$B_n = \pi_1\left(\frac{\mathbb{C}^n - \{\text{hyperplanes}\}}{S_n}\right)$$

The elements of B_n can be imagined as braids



More generally, for a complex reflection group $G \subseteq GL(V)$, we want to study

$$\pi_1\left(\frac{V - \{\text{hyperplanes}\}}{G}\right)$$

Relation with Representation Theory

Hecke algebras are immensely important to representation theory. The Hecke algebra of a complex group reflection W acting on a $\mathbb{C}$-vector can be specified as

$$\mathcal{H}_q(W) \cong \pi_1\left(\frac{V - \{hyperplanes\}}{W}\right) \Big/ \{Relations\}$$

Thus, understanding the braid groups gives a better understanding of the Hecke algebra.
As an example, one has Knizhnik-Zamolodchikov functor

$$\begin{tikzcd} \{\mathbb{C}[W]\text{-modules}\} \arrow[r, no head, "\text{Induction}"'] \arrow[rr, bend left, dotted, no head, "\text{Deform}"] & \{\text{RCA-modules}\} \arrow[r, no head, "\text{KZ}"'] & \{\text{Hecke-modules}\} \end{tikzcd}$$

One obtains representations of $\mathcal{H}_q(W)$ from representations of W by computing monodromy along the paths in the fundamental group.