Hecke Algebra Representations through the KZ-Functor

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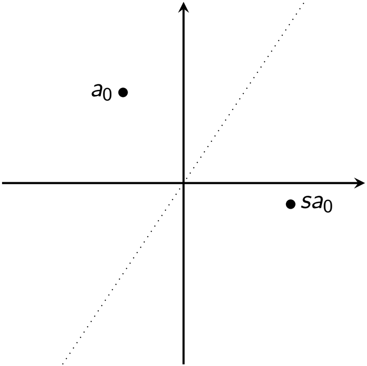
Supervised by Dr. Yaping Yang and Prof. Arun Ram
The University of Melbourne

Complex Reflection Groups



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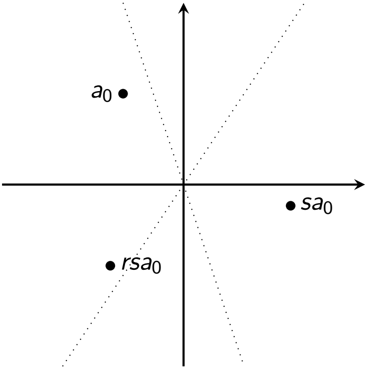

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The composition of these two symmetries of the plane is called rs

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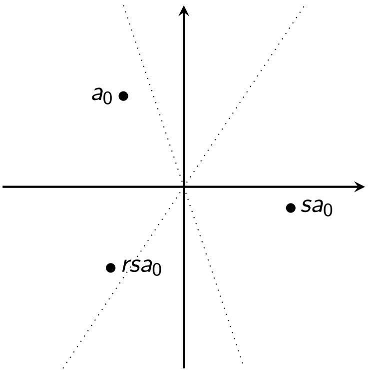

Imagine a reflection s across a hyperplane in a complex vector space.
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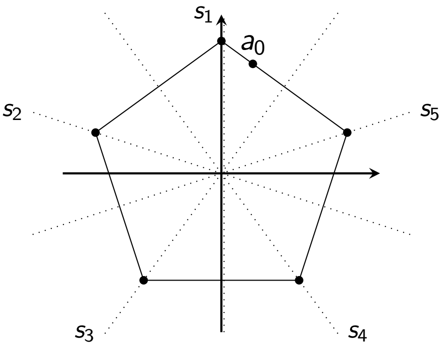
The composition of these two symmetries of the plane is called rs
A collection of transformations of a space, each generated by a common set of reflections is called a complex reflection group

Symmetries of a Pentagon



1. Start with a basepoint a_0.

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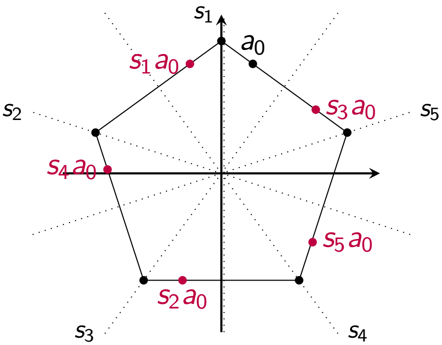

1. Start with a basepoint a_0.
2. Reflect across the reflection axes a_0.

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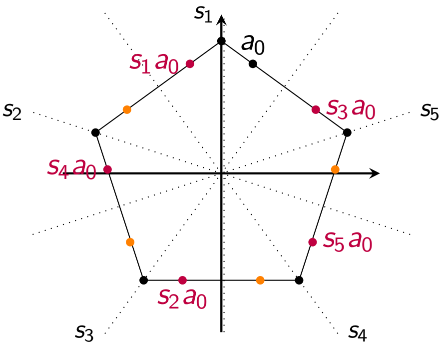

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3. Reflect all the reflections of a_0 across the axes.

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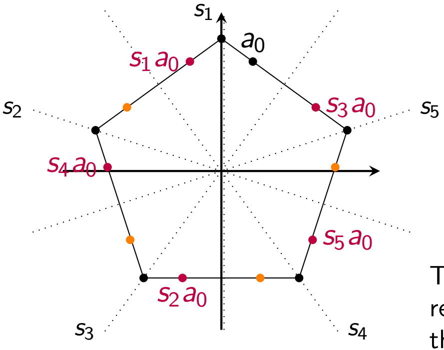

1. Start with a basepoint a_0.
2. Reflect across the reflection axes a_0.
3. Reflect all the reflections of a_0 across the axes.

These symmetries are a complex reflection group. The ten points are the orbit of a_0.

Doing the same for each n-gon, we obtain the dihedral groups.

Algebras and Modules

An algebra is a vector space with a multiplication. For example, $End(V)$ for vector space V.

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A module over an algebra is a homomorphism from the algebra to $End(E)$ where E is some complex vector space.

A module over the group algebra is called a W_{fin}-module.

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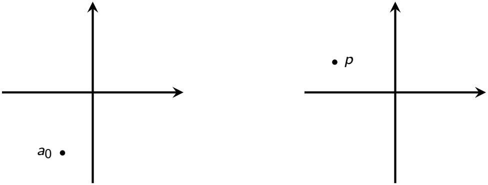
We will mainly discuss the conversion from W_{fin}-modules to Hecke-modules

Monodromy

Imagine two flies on two different spaces, and the direction and speed of the second fly depends on the positions of both flies.



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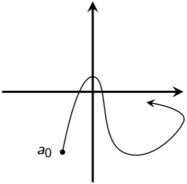
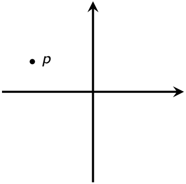
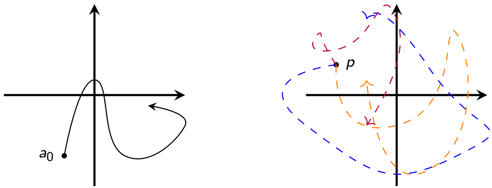
Imagine two flies on two different spaces, and the direction and speed of the second fly depends on the positions of both flies. Suppose the first fly begins to move along a continuous path.





Monodromy

Imagine two flies on two different spaces, and the direction and speed of the second fly depends on the positions of both flies. Suppose the second fly is standing at some point, and the first fly begins to move along a continuous path.



How does the second fly move as a function of the first's position?

More precisely, the first space is meant to be the vector space V on which the complex reflection group acts by reflections, and the second is supposed to be any representation E. If the position in the first space is written as $(x_1, \cdots, x_N)$, and the position in the second space is written as $(f_1, f_2, \cdots, f_d)$, the motion is described by a system of differential equations

$$\frac{\partial f_i}{\partial x_j} = \varphi_{i,j}\left(x_1, x_2, \cdots, x_N, f_1, f_2, \cdots, f_d\right)$$

where $i \in \{1, 2, \cdots, d\}$ and $j \in \{1, 2, \cdots, N\}$, with some initial condition to specify where the flies start.

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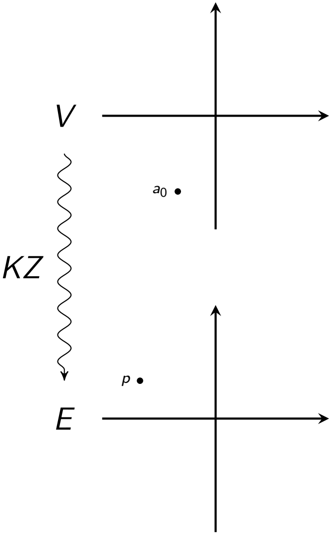
The differential equations come from the rational Cherednik algebra of the group. It is dependent on the representation E being used, and a complex parameter for each conjugacy class of reflections in the group.

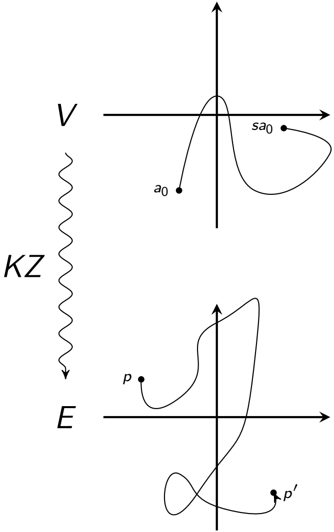
Deforming the Complex reflection Group

Recall a complex reflection group is composed of some transformation of the space. One can deform the transformations using monodromy described above to get a new collection of transformations.

A complex reflection group W acts by reflections on some space V. It can also transform another E, but for simplicity we can take $E = V$ with the same reflection actions. We will deform the transformations on E to get a Hecke-module.

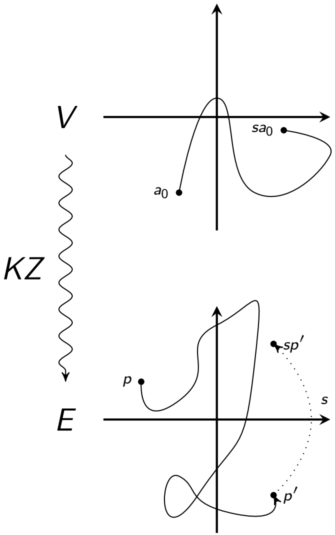


Let p be a point in E, and choose a basepoint $a_0 \in V$. We will define the transformed action T_s on p.



Let p be a point in E, and choose a basepoint $a_0 \in V$. We will define the transformed action T_s on p.

The rational Cherednik algebra gives us a rule for how a path from a_0 to sa_0 in V corresponds to a path from p to some endpoint p' in E.



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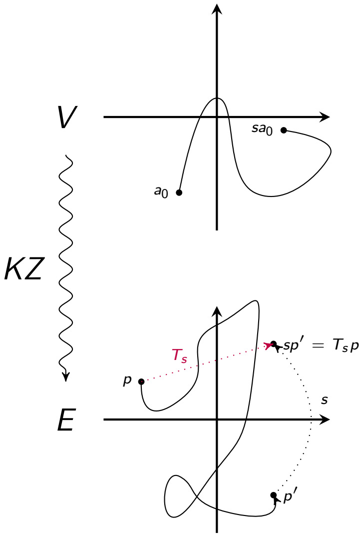

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Act upon p' by s

This final point will be $T_s p$, the deformed action for the Hecke algebra

Case of Odd Dihedral Groups

If $s_0, s_2, \cdots, s_{n-1}$ are the reflections of D_n, then the irreducible representations are ρ_k with $0 < j < n/2$ and

$$\rho_j(s_k) = \begin{pmatrix} 0 & e^{2\pi ji/n} \\ e^{-2\pi ji/n} & 0 \end{pmatrix}$$

All reflections are conjugate, so there will be a single complex parameter $c_0 \in \mathbb{C}$ in the differential equations.
Recall that the differential equations are of the form

$$\frac{\partial f_i}{\partial x_j} = \varphi_{i,j}\left(x_1, x_2, \cdots, x_N, f_1, f_2, \cdots, f_d\right)$$

Case of Odd Dihedral Groups

$$\frac{\partial f_1}{\partial x_1} = \frac{nc_0}{x_1^n - x_2^n}\left(-x_1^{n-1}f_1 + x_2^{n-1}\left(\frac{x_1}{x_2}\right)^{j-1} f_2\right)$$

$$\frac{\partial f_1}{\partial x_2} = \frac{nc_0 x_2^{n-1}}{x_1^n - x_2^n}\left(f_1 - \left(\frac{x_1}{x_2}\right)^{j} f_2\right)$$

$$\frac{\partial f_2}{\partial x_1} = \frac{nc_0 x_1^{n-1}}{x_1^n - x_2^n}\left(\left(\frac{x_2}{x_1}\right)^{j} f_1 - f_2\right)$$

$$\frac{\partial f_2}{\partial x_2} = \frac{nc_0}{x_1^n - x_2^n}\left(-x_1^{n-1}\left(\frac{x_2}{x_1}\right)^{j-1} f_1 + x_2^{n-1} f_2\right)$$

Case of Symmetric Group

The irreducible representations of S_n are quite intractable, so we work with just the alternating, permutation and regular representations. The generating reflections are all conjugate, so there is again a single complex parameter $c_0 \in \mathbb{C}$.

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For the alternating representation, the differential equations are

$$\frac{\partial f}{\partial x_k} = 2c_0 f \sum_{\ell \neq k} \frac{1}{x_\ell - x_k}$$

where $k \in \{1, 2, \cdots, n\}$.

In the case of the permutation representation, it is

$$
\begin{aligned}
\frac{\partial f_i}{\partial x_k} &= \frac{c_0}{x_k - x_i}\left(-f_i + f_k\right) \\
\frac{\partial f_i}{\partial x_i} &= c_0 \sum_{\ell \neq i} \frac{1}{x_i - x_\ell}\left(-f_i + f_\ell\right)
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where $i, k \in \{1, 2, \cdots, n\}, i \neq k$.

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where $i, k \in \{1, 2, \cdots, n\}, i \neq k$.
And for the regular representation,

$$\frac{\partial f_\sigma}{\partial x_k} = -c_0 f_\sigma \left[\sum_{\ell=1,\ell\neq k}^{n} \frac{1}{x_k - x_\ell}\right] + c_0 \left[\sum_{\ell=1,\ell\neq k}^{n} \frac{f_{(\ell,k)\sigma}}{x_k - x_\ell}\right]$$

with $k \in \{1, 2, \cdots, n\}$, $\sigma \in S_n$.

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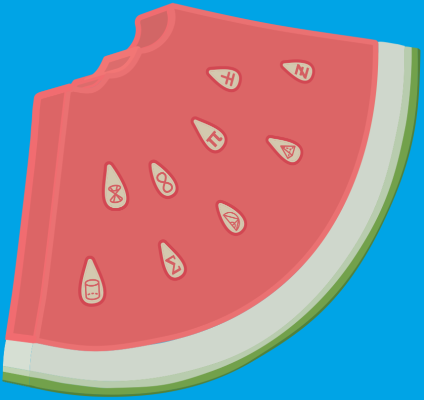

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