Representation Theory Student Seminar

Peter-Weyl Theorem II.

Recall that for finite group G, and representation (π, V), we define

$$\chi_V : G \longrightarrow \mathbb{C}$$
$$g \longmapsto Tr(\pi(g)).$$

We had the following fact:

Theorem: For function $f, g \in \mathbb{C}[G]$ define

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{g \in G}' f(g) \overline{g(g)}.$$

It holds that for irreducible representation (π, V), (ρ, W) of G,

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & , \text{if } V \cong W \\ 0 & , \text{if } V \ncong W. \end{cases}$$

Moreover, $\{ \chi_V \mid V \text{ irreducible representation} \}$ of G forms an orthonormal basis of $\mathbb{C}[G]$.

Today, we generalize this to compact Lie group by studying the structure of $C_{alg}(G)$.

Throughout, G a compact Lie group
$C(G)$ the cts $\mathbb{C}$-valued functions on G
$L^2(G)$ the square integrable $\mathbb{C}$-valued functions on G.
$C_{alg}(G)$ - The algebra of matrix coefficients

Recall that for representation P of G and ~~$\xi \in P^*$, $v \in P$, we define~~ $\eta, \xi \in P$, define

$$f_{P,\eta,\xi} : \begin{array}{ccc} G & \longrightarrow & \mathbb{C} \\ g & \longmapsto & \langle \xi, g\eta \rangle . \end{array}$$

This is called a matrix coefficient of G and we write

$$C_{alg}(G) = \left\{ f_{P,\eta,\xi} \;\middle|\; \begin{array}{l} P \text{ a representation of } G \\ \eta, \xi \in P \end{array} \right\}.$$

For representation V of G, write

$$V^{fin} = \left\{ v \in V \;\middle|\; \begin{array}{l} v \text{ contained in finite dimensional} \\ \text{subrepresentation of } V \end{array} \right\}.$$

From last time, we have

$$C_{alg}(G) = C(G)^{fin}.$$

<u>Theorem</u> (Peter-Weyl): $C_{alg}(G)$ is dense in $C(G)$ with respect to the topology of uniform convergence.

Theorem: There is an isomorphism of $G \times G$ representations

$$C_{alg}(G) \cong \bigoplus \bar{P} \otimes P$$ ($\bigoplus$ goes over all irreps P)

given by $\eta \otimes \xi \longmapsto f_{P,\eta,\xi}$.

(ii) Each $\bar{P} \otimes P$ is an irreducible representation of $G \times G$

(iii) If we define for each irrep P

$$\langle \cdot , \cdot \rangle : (\bar{P} \otimes P) \times (\bar{P} \otimes P) \longrightarrow \mathbb{C}$$

by extending

$$\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle = \frac{\langle \eta_1, \eta_2 \rangle \langle \xi_1, \xi_2 \rangle}{\dim P}.$$

then the isomorphism $\bigoplus \bar{P} \otimes P \cong C_{alg}(G)$ preserves the inner product ($C_{alg}(G)$ endowed with usual $L^2(G)$-inner product).

<u>Proof</u> (i) Recall $C_{alg}(G) = C(G)^{fin}$ and by defn

$$C(G)^{fin} = \text{Sum of irreps in } C(G).$$

For irrep P, we have

$$C(G)_P = P \otimes \mathrm{Hom}_G(P, C(G))$$

(by identifying $v \otimes f \longmapsto f(v)$)

So then

$$C_{alg}(G) = C(G)^{fin} = \bigoplus_P C(G)_P$$

$$= \bigoplus_P \bar{P} \otimes \mathrm{Hom}_G(\bar{P}, C(G)).$$

But see that the map

$$\Phi : P \longrightarrow \mathrm{Hom}_G(\bar{P}, C(G))$$
$$v \longmapsto (\xi \longmapsto (g \longmapsto \underbrace{\xi(g^{-1}v)}_{\cancel{\langle \xi, g\cdot v\rangle}\ \langle v, g\xi\rangle}))$$

<u>Claim</u>: Φ is G-equivariant, i.e if $g \in G$, $v \in P$ then $\Phi(g\cdot v) = g\cdot(\Phi(v))$ (where the action of G on $\mathrm{Hom}_G(-,-)$ is defined in the usual way).
Moreover, Φ is an isomorph.

$$P \cong \mathrm{Hom}_G(\bar{P}, C(G)).$$

Injectivity: Use Schur lemma to prove $\ker \Phi = \{0\}$ (P is irreducible)

Surjectivity: If $f \in \mathrm{Hom}_G(\bar{P}, C(G))$ so $f : \bar{P} \longrightarrow C(G)$, we have linear map

$$\begin{aligned} P &\longrightarrow \mathbb{C} \\ \eta &\longmapsto (f(\eta))(1) \end{aligned}$$

So there exists $\xi \in P$ such that

$$\langle \eta, \xi \rangle \langle \xi, \eta \rangle = (f(\eta))(1)$$

for every $\eta \in P$. Then it is true that $\Phi(\xi) = f$. Because

$$\begin{aligned} (\Phi(\xi)(\eta))(g) &= \langle \eta, g \cdot \xi \rangle \\ &= \langle g^{-1}\eta, \xi \rangle \\ &= (f(g^{-1}\eta))(1). \\ &\overset{?}{=} g^{-1} f(\eta)(1) \\ &= f(\eta)(g) \end{aligned}$$

So $\Phi(\xi)(\eta) = f(\eta)$ at all $g \in G$
So $\Phi(\xi) = f$
So $f \in \mathrm{Im}\,\Phi$
So Φ is surjective

This proves isomorphism

So now

$$C_{alg}(G) = \bigoplus \bar{P} \otimes \mathrm{Hom}_G(\bar{P}, C(G))$$
$$= \bigoplus \bar{P} \otimes P.$$

Which is (i).

Now to prove (ii). Notice that

$$\mathrm{End}_{G_1 \times G_2}(P_1 \otimes P_2) \cong \mathrm{End}_{G_1}(P_1) \otimes \mathrm{End}_{G_2}(P_2)$$
$$\cong \mathbb{C} \otimes \mathbb{C} \quad \left(\text{irreducibility of } P_1, P_2\right)$$
$$\cong \mathbb{C}$$

So $P_1 \otimes P_2$ is irreducible (otherwise project onto invariant subspace is non-homothetic element of $\mathrm{End}_{G_1 \times G_2}(P_1 \otimes P_2)$.

Finally to show (iii).

Pull-back of $L^2(G)$ inner product on $C_{alg}(G)$ to $\bigoplus \bar{P} \otimes P$.

Claim: This will be $G \times G$ invariant innerproduct on $\bigoplus \bar{P} \otimes P$.

By Schur's lemma, the summands $\overline{P}\otimes P$ are orthogonal w.r.t. this inner product and there exists a unique choice of inner product on each $\overline{P}\otimes P$ (up to homotethy). So we will have for each P

$$\langle\cdot,\cdot\rangle_P : \overline{P}\otimes P \times \overline{P}\otimes P \longrightarrow \mathbb{C}.$$

$$(\eta_1\otimes\xi_1, \eta_2\otimes\xi_2) \longmapsto K_P \overline{\langle\eta_1,\eta_2\rangle}\langle\xi_1,\xi_2\rangle$$

for some $K_P\in\mathbb{C}$ $\quad \langle f_{P,\eta_1,\xi_1}, f_{P,\eta_2,\xi_2}\rangle = \int_G f_{\dots}\overline{f_{\dots}}\,dg$

(See $\left((g,h)(\eta_1\otimes\xi_1), (g,h)(\eta_2\otimes\xi_2)\right) = \left(g\eta_1\otimes h\xi_1, g\eta_2\otimes h\xi_2\right)$

$$\longmapsto K_P\overline{\langle g\eta_1, g\eta_2\rangle}\langle h\xi_1, h\xi_2\rangle$$

$$= K_P\overline{\langle\eta_1,\eta_2\rangle}\langle\xi_1,\xi_2\rangle$$

$$= \langle\eta_1\otimes\xi_1, \eta_2\otimes\xi_2\rangle.$$

i.e. unitary)

We need to compute K_P.

Let $\{e_i\}_{i=1}^n$ be orthonormal basis of P. Set

$$M_{ij}(g) = \langle e_i, g e_j\rangle = f_{P,e_i,e_j}(g).$$

So $M_{ij}\in C_{alg}(G)$.

Then we have

$$\int_G \overline{M_{ij}(g)} M_{ke}(g)\,dg = \cancel{\int_G M_{ji}(g^{-1}) M_{ke}(g)\,dg}$$

$$= \cancel{K_p} \int_G \overline{f_{P,e_i,e_j}(g)}\, f_{P,e_k,e_\ell}(g)\,dg$$

$$= \langle e_i \otimes e_j, e_k \otimes e_\ell \rangle$$

$$= K_p \langle e_i, e_k \rangle \overline{\langle e_j, e_\ell \rangle}$$

$$= K_p \delta_{ik} \cancel{\delta_{ke}} \delta_{j\ell}.$$

Taking $i=k$ and sum from $i=1$ to $n = \dim P$

$$\sum_{i=1}^n \int_G \overline{M_{ij}(g)} M_{i\ell}(g)\,dg = \int_G \sum_{i=1}^n M_{ji}(g^{-1}) M_{i\ell}(g)\,dg$$

$$= \int_G M_{j\ell}(I)\,dg$$

$$= \delta_{j\ell}.$$

But also

$$\sum_{i=1}^n \int_G \overline{M_{ij}(g)} M_{i\ell}(g)\,dg = \sum_{i=1}^n K_p \delta_{ii} \delta_{j\ell}$$

$$= K_p \dim P\, \delta_{j\ell}.$$

If $j=\ell$ then $K_p \dim P = 1$ so $K_p = \dfrac{1}{\dim P}$.

Now to prove

<u>Corollary</u>: (i) Any finite dim. rep of G is determined upto isomorphism by its character

(ii). We have orthogonality:

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & , \text{if } V \cong W \\ 0 & , \text{if } V \not\cong W \end{cases}$$

(for V, W irreducible).

(iii) The characters are an orthonormal basis for the space of class functions.

<u>Proof</u> (ii). If P irrep of G then

$$\chi_P = \Phi\left(\sum_{i=1}^{\dim P} \bar{e}_i \otimes e_i\right) := \sum_{i=1}^{\dim P} f_{P, e_i, e_i}$$

So χ_P, χ_Q are orthogonal for $P \not\cong Q$ because

$$\sum_{i=1}^{\dim P} \bar{e}_i \otimes e_i \quad \text{and} \quad \sum_{j=1}^{\dim Q} \bar{c}_i \otimes c_i$$

are in $\bigoplus \bar{P} \otimes P$.

Same idea for normality:

$$\langle \chi_P, \chi_P \rangle = \left\langle \sum_{i=1}^{n} \bar{e}_i \otimes e_i , \sum_{j=1}^{n} \bar{e}_j \otimes e_j \right\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \bar{e}_i \otimes e_i , \bar{e}_j \otimes e_j \rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\dim P} \overline{\langle e_i, e_j \rangle} \langle e_i, e_j \rangle .$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\delta_{ij}}{\dim P}$$

$$= 1 .$$

Proof of (i). Follows from (ii). Because if $V = W$ then $\chi_V = \chi_W$ (trivial) and if $V \neq W$, then we have $\langle \chi_V, \chi_W \rangle = 0$ so $\chi_V \neq \chi_W$ (cleaning $\langle \chi_V, \chi_W \rangle > 0$ -).

Proof of (iii). ~~We~~ The algebraic class functi from $\bar{P} \otimes P$ are the fixed points

$$(\bar{P} \otimes P)^G \cong \mathrm{Hom}_G(\bar{P}, P)$$

$$\cong \mathrm{Hom}_G(P, P)$$

$$\cong \mathbb{C} \quad \text{(Schur)}$$

Since $\sum_{i=1}^{n} \bar{e}_i \otimes e_i \in (\bar{P} \otimes P)^G$, we have

$$(\bar{P} \otimes P)^G = \mathbb{C}\text{-span}\{\textstyle\sum \bar{e}_i \otimes e_i\} = \mathbb{C}\text{-span}\{\chi_P\}.$$