Localisation of Rings & Local Properties

Motivation: The construction of $\mathbb{Q}$.

Take $S = \mathbb{Z} - \{0\}$, which is closed under multiplication as $\mathbb{Z}$ is an integral domain.

Given $(a,b),(c,d) \in \mathbb{Z} \times S$, say

$$(a,b) \sim (c,d) \text{ iff } ad - bc = 0$$

$$\overset{\shortparallel}{\tfrac{a}{b}} \qquad \overset{\shortparallel}{\tfrac{c}{d}}$$

The ordered pairs of $\mathbb{Z} \times S$ are interpreted as fractions with denominators in S. The operations are

$$(a,b) + (c,d) := (ad + bc, bd)$$

$$(a,b) \cdot (c,d) := (ac, bd)$$

This works well with the equivalence relation $\sim$, so $\mathbb{Q}$ is the set of equivalence classes

$$\mathbb{Q} = \mathbb{Z} \times S / \sim$$

with operations

$$[(a,b)] + [(c,d)] = [(a,b) + (c,d)]$$

$$[(a,b)] \cdot [(c,d)] = [(ac, bd)]$$

and additive and multiplicative identities

$$1_{\mathbb{Q}} = [(1,1)] \qquad 0_{\mathbb{Q}} = [(0,1)].$$

This makes $\mathbb{Q}$ a field (in particular it is a commutative ring like $\mathbb{Z}$)

Generalise to all commutative rings.

From now, A is a commutative ring.

Definition: A set $S \subset A$ is multiplicatively closed if
$$1_A, \cancel{0} \in S \quad \text{and} \quad \forall x,y \in A,\ xy \in A.$$

Given $(a,s),(b,t) \in A \times S$, say
$$(a,s) \sim (b,t) \text{ iff } \exists u \in S \text{ s.t. } u(at - bs) = 0$$

Proposition: $\sim$ is an equivalence relation.

Proof: Symmetry and reflexivity are easy. We prove only transitivity here.

~~Let $(a,s),(b,t),(c,r) \in A \times S$ with $(a,s) \sim (b,t)$, so $u(at-bs)=0$ for some $u \in S$ $(b,t) \sim (c,$~~

Let $(a,r),(b,s),(c,t) \in A \times S$ with

$(a,r) \sim (b,s)$, with $u(as - br) = 0$ for some $u \in S$

$(b,s) \sim (c,t)$, with $v(bt - cs) = 0$ for some $v \in S$.

then,
$$\begin{aligned}
(uvs)(at - cr) &= (uv)((as)t - (\cancel{b}cs)r) \\
&= (uv)((as)t - (br)t + (bt)r - (cs)r) \\
&= (vt)(u(as - br)) + (ur)(v(bt - cs)) \\
&= (vt)(0) + (ur)(0) \\
&= 0
\end{aligned}$$

Since $uvs \in S$ (multiplicative closure), we have that $(a,r) \sim (c,t)$ as desired. $\square$

We also have

<u>Proposition</u>: Let $(a,s) \sim (a',s')$, and $(b,t) \sim (b',t')$.
Then,

$$(at+bs, st) \sim (a't'+b's', s't')$$
$$(ab, st) \sim (a'b', s't')$$

<u>Proof</u>: Let $u,v \in S$ be such that
$$u(as'-a's)=0 \qquad v(bt'-b't)=0$$
Then,
$$uv\left[(at+bs)(s't')-(a't'+b's')(st)\right]$$
$$\cancel{=(uv)[(ss')]}$$
$$=(uv)\left[(tt')(as'-a's)+(ss')(bt'-b't)\right]$$
$$=(vtt')\left(u(as'-a's)\right)+(uss')\left(v(bt'-b't)\right)$$
$$=0$$

$$uv(abs't'-a'b'st)$$
$$=uv(abs't'-a'bst'+a'bst'-a'b'st)$$
$$=uv\left((bt')(as'-a's)+(a's)(bt'-b't)\right)$$
$$=(vbt')\left(u(as'-a's)\right)+(ua's)\left(v(bt'-b't)\right)$$
$$=0$$

$\square$

This allows us to define .

<u>Definition</u>: $S^{-1}A$ is the set of equivalence classes

$$S^{-1}A = A \times S /\sim$$

with operations

$$[(a,s)] + [(b,t)] = [(at+bs, st)]$$

$$[(a,s)] \cdot [(b,t)] = [(ab, st)]$$

This makes $S^{-1}A$ a commutative ring with identities

$$1_{S^{-1}A} = [(1,1)] \qquad 0_{S^{-1}A} = [(0,1)] .$$

From now, we write a/s for (a,s), and omit $[\cdot]$.

~~If~~ We have the property:

<u>Proposition</u>: Let $f: A \longrightarrow B$ be a ring homomorphism. (where
~~They~~ such that $f(s) \in B$ is a unit $\forall s \in S \subseteq A$
S is multiplicatively closed. Then there is a unique

$$S^{-1}f : S^{-1}A \longrightarrow B$$

with

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & S^{-1}A \arrow[d, "S^{-1}f"] \\ & B \end{tikzcd} \qquad \left(i: A \longrightarrow S^{-1}A,\ a \longmapsto a/1 \right).$$

Proof: (Existence) Define

$$S^{-1}f : S^{-1}A \longrightarrow B$$
$$\frac{a}{s} \longmapsto f(a)f(s)^{-1}$$

This is well-defined: if $\frac{a}{s} = \frac{b}{t}$, then $0 = u(at-bs)$ for some $u \in S$ and hence,

$$0 = f(0) = f(u(at-bs)) = f(u)(f(a)f(t) - f(b)f(s))$$
$$f(a)f(t) = f(b)f(s)$$
$$f(a)f(s)^{-1} = f(b)f(t)^{-1}.$$

Moreover, $S^{-1}f$ is a homomorphism:

$$S^{-1}f\left(\frac{a}{s} + \frac{b}{t}\right) = S^{-1}f\left(\frac{at+bs}{st}\right)$$
$$= f(at+bs)f(st)^{-1}$$
$$= f(a)f(t)f(s)^{-1}f(t)^{-1} + f(b)f(s)f(s)^{-1}f(t)^{-1}$$
$$= f(a)f(s)^{-1} + f(b)f(t)^{-1}$$
$$= S^{-1}f(a/s) + S^{-1}f(b/t)$$

$$S^{-1}f\left(\frac{a}{s} \cdot \frac{b}{t}\right) = f(ab)f(st)^{-1}$$
$$= f(a)f(s)^{-1}f(b)f(t)^{-1}$$
$$= S^{-1}f(a/s)\, S^{-1}f(b/t).$$

and for all $a \in A$,

$$(S^{-1}f \circ \iota)(a) = S^{-1}f(a/1) = f(a)f(1)^{-1} = f(a)1_B^{-1} = f(a)$$

So $S^{-1}f \circ \iota = f$. This proves existence.

(Uniqueness). If such $S^{-1}f : S^{-1}A \longrightarrow B$ exists, then,

$$S^{-1}f(a/s) = S^{-1}f(as^{-1}) = S^{-1}f(a)S^{-1}f(s)^{-1}$$
$$= (S^{-1}f\circ\iota)(a)(S^{-1}f\circ\iota)(s)^{-1}$$
$$= f(a)f(s)^{-1}.$$

So $S^{-1}f : S^{-1}A \longrightarrow B$ is uniquely specified. $\square$

The inclusion $\iota : A \longrightarrow S^{-1}A$ satisfies

- $\forall s\in S$, $\iota(s)$ is a unit
- if $\iota(a) = 0 \Rightarrow as = 0$ for some $s\in S$
- For every $x\in S^{-1}A$, there exist $a\in A, s\in S$ with

$$x = \iota(a)\iota(s)^{-1}.$$

Conversely,

<u>Proposition</u>: Let $g : A \longrightarrow B$ satisfy

(i). $g(s)\in B$ is a unit $\forall s\in S$.

(ii). $g(a) = 0$ then $as = 0$ for some $s\in S$.

(iii) $\forall b\in B$, $\exists a\in A\ \exists s\in S$ s.t. $b = g(a)g(s)^{-1}$.

Then $B \cong S^{-1}A$.

<u>Proof</u>: Define $\phi : S^{-1}A \longrightarrow B$ by

$$\phi(a/s) = g(a)g(s)^{-1}.$$

It is easy to check ϕ is a well-defined homomorphism. By (iii), ϕ is surjective. Moreover, If $\phi(a/s) = 0$, then,

$$0 = \phi(a/s) = g(a)g(s)^{-1}$$
$$g(a) = 0.$$

By (ii), $as^* = 0$ for some $s^*\in S$. This means

$$a = \frac{a}{1} = 0$$

So $\ker\phi = \{0\}$, which makes ϕ injective as well. $\square$

Examples

Ex.1: Let $P \subseteq A$ prime. Then $S = A - P$ is multiplicatively closed. Define

$$A_P := S^{-1}A = (A-P)^{-1}A \quad \text{(localisation at } P\text{)}.$$

Claim: A_P is a local ring.

For every $s,t \in A-P$, $s/t \in A_P$ is a unit. If an ideal contains a unit, then it is all of A, so any ideal is a subset of

$$m = \{ p/s \mid p \in P,\ s \notin P \} \subseteq S^{-1}A .$$

Moreover, m is an ideal; which makes it the unique maximal ideal of $S^{-1}A$.

Ex.2: Let $f \in A$, set

$$S = \{ f^n \mid n \geq 0 \}.$$

Write $A_f = S^{-1}A$.

Ex.3. For any ideal $I \subseteq A$, $1_A + I \subseteq A$ is multiplicatively closed:

If $x+1, y+1 \in 1+I$ for any $x,y \in I$,

$$(x+1)(y+1) = \underbrace{(x+y+xy)}_{\in I} + 1 \in 1+I .$$

Ex.4 For any $n \geq 1$, take $S = \{1, n, n^2, n^3, \dots\}$. Then, $S^{-1}\mathbb{Z}$ is the rationals whose denominator is a power of n. E.g. for $n=2$, we have the dyadic rationals.

Extension to A-mod.

For $S \subseteq A$ multiplicatively closed; M an A-module, ~~Say~~ given

$$\frac{m_1}{s_1} := (m_1, s_1), \frac{m_2}{s_2} := (m_2, s_2) \in M \times S,$$

say

$$\frac{m_1}{s_1} \sim \frac{m_2}{s_2} \text{ iff } \exists u \in S,\ u(s_2 m_1 - s_1 m_2) = 0.$$

We have an $S^{-1}A$-module denoted $S^{-1}M$ as the equivalence classes

$$S^{-1}M = M \times S / \sim$$

with

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}$$

$$\frac{r}{s} \cdot \left(\frac{m}{t}\right) = \frac{rm}{st}.$$

$$0_{S^{-1}M} = \frac{0}{1}$$

This is the module $S^{-1}M$ over the ring $S^{-1}A$.

Given A-mod homomorphism

$$f : M \longrightarrow N,$$

we have $S^{-1}A$-mod homomorphism

$$S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$$

$$\frac{m}{s} \longmapsto \frac{f(m)}{s}.$$

Moreover, given,

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & N \arrow[r, "g"] & P \end{tikzcd}$$

we have

$$\begin{tikzcd} S^{-1}M \arrow[r, "S^{-1}f"] \arrow[rr, bend left, "S^{-1}(g\circ f)"] & S^{-1}N \arrow[r, "S^{-1}g"] & P \end{tikzcd}$$

It turns out $S^{-1}(g\circ f) = S^{-1}g \circ S^{-1}f$. Because, by def.

$$S^{-1}(g\circ f) : S^{-1}M \longrightarrow S^{-1}P$$
$$\frac{m}{s} \longmapsto \frac{(g\circ f)(m)}{s}$$

But,

$$\cancel{S^{-1}(g\circ f)\left(\frac{m}{s}\right) = \frac{1}{s}(g\circ f)(m) = \frac{1}{s}g(f(m))}$$
$$\cancel{= g\left(\frac{1}{s}f(m)\right)}$$
$$\cancel{= g\left(\frac{f(m)}{s}\right) = g\left(S^{-1}f(m/s)\right)}$$
$$\cancel{=}$$

$$S^{-1}(g\circ f)(m/s) = \frac{(g\circ f)(m)}{s} = \frac{g(f(m))}{s}$$
$$= S^{-1}g\left(\frac{f(m)}{s}\right) = S^{-1}g\left(S^{-1}f(m/s)\right)$$
$$= (S^{-1}g\circ S^{-1}f)(m/s).$$

This means localisation is a <u>functor</u>

$$S^{-1} : A\text{-mod} \longrightarrow S^{-1}A\text{-mod}.$$

acting on objects as $M \longmapsto S^{-1}M$ and morphisms as $f \longmapsto S^{-1}f$.

Even more:

<u>Proposition</u>: Localisation is an exact functor.

I.e. let

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

be exact at M ($\ker g = \operatorname{Im} f$). Then

$$S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$$

is exact at $S^{-1}M$.

<u>Proof</u>: Since $g \circ f = 0$, we have

$$0 = S^{-1}(g \circ f) = S^{-1}g \circ S^{-1}f$$

So $\operatorname{Im} S^{-1}f \subseteq \ker S^{-1}g$.

Conversely, let $\frac{m}{s} \in \ker S^{-1}g$. Then,

$$0 = S^{-1}g(m/s) = \frac{g(m)}{s}$$

So $\exists u \in S$ s.t. $ug(m) = 0$. Then

$$0 = ug(m) = g(um)$$

So $um \in \ker g = \operatorname{Im} f$. Let $um = f(m^*)$. Then,

$$\frac{m}{s} = \frac{um}{us} = \frac{f(m^*)}{us} = S^{-1}f\left(\frac{m^*}{us}\right) \in \operatorname{Im} S^{-1}f.$$

So also $\ker S^{-1}g \subseteq \operatorname{Im} S^{-1}f$.

So the sequence

$$S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$$

is exact at $S^{-1}M$. $\square$

Corollary: Let ~~M~~ $N \subseteq M$ be A-modules. Then,

$$S^{-1}M / S^{-1}N \cong S^{-1}(M/N).$$

Proof: Let $\iota: N \hookrightarrow M$ be the inclusion,

$$q: M \longrightarrow M/N,$$
$$m \longmapsto m + N$$

the quotient map. Clearly, ι is injective and q is surjective so

$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{q} M/N \longrightarrow 0$$

is exact at N and M/N. Also,

$m \in \ker q$ iff $m + N = N$ iff $m \in N$ iff $m \in \operatorname{Im} \iota$

So sequence is also exact at M. Then we have the exact sequence

$$0 \longrightarrow S^{-1}N \xrightarrow{S^{-1}\iota} S^{-1}M \xrightarrow{S^{-1}q} S^{-1}(M/N) \longrightarrow 0$$

Hence,

$$\begin{aligned} S^{-1}(M/N) &= \operatorname{Im}(S^{-1}q) \\ &\cong S^{-1}M / \ker S^{-1}q \quad \text{(First isomorphic theorem)} \\ &= S^{-1}M / \operatorname{Im} \iota \\ &= S^{-1}M / S^{-1}N. \end{aligned}$$

$\square$

Local Properties

A property of an A-module is a local property if being true of any A-module M is equivalent to being true of any localisation M_p at primes $p \subseteq A$.

The property of being a zero-module is local:

Proposition: The following are equivalent:

(i). ~~$M \neq 0$~~ $M = 0$

(ii) $M_p = 0$ for all prime $p \subseteq A$.

(iii) $M_m = 0$ for all maximal ~~ideal~~ $m \subseteq A$.

Proof:

(i) $\Longrightarrow$ (ii) Obvious

(ii) $\Longrightarrow$ (iii) Maximal ideals are prime

(iii) $\Rightarrow$ (i). Suppose for contradiction that $M_m = 0$ for every maximal $m \subseteq A$, but $M \neq 0$. Let $x \in M$, $x \neq 0$.

The annihilator

$$\mathrm{Ann}(x) = \{a \in A \mid ax = 0\} \subseteq A$$

is an ideal. Since

$$1 \cdot x = x \neq 0$$

$1 \notin \mathrm{Ann}(x)$ so $\mathrm{Ann}(x) \neq A$. Let $m \subseteq A$ be a maximal ideal with $\mathrm{Ann}(x) \subseteq m$. Then,

$$\frac{x}{1} = 0 \quad \text{as } M_m = 0.$$

Then ~~$\exists x \in S$ with $ux = 0$~~, so $\exists u \in S$ (where $S = A - m$) with $ux = 0$. So $u \in \mathrm{Ann}(x) \subseteq m$.

This contradicts $u \in S = A - m$.

$\square$

<u>Proposition</u>: Injectivity is local. That is, for

$$f : M \longrightarrow N,$$

f is injective iff

$$f_p : M_p \longrightarrow N_p$$

is injective where

$$f_p = (A-p)^{-1} f \quad M_p = (A-p)^{-1} M \quad N = (A-p)^{-1} N$$

for every prime $p \subseteq A$.

<u>Proof</u>: If $f : M \longrightarrow N$ injective, then

$$0 \longrightarrow M \xrightarrow{f} N$$

is exact at M. Then for any $p \subseteq A$ prime,

$$0 \longrightarrow M_p \xrightarrow{f_p} N$$

is ~~exat~~ exact, so f_p is injective.

Conversely, assume $f_p : M_p \longrightarrow N_p$ is injective for every prime $p \subseteq A$. In particular,

$$f_m : M_m \longrightarrow N_m$$

is injective for every maximal $m \subseteq A$.

We have the exact sequence

$$0 \longrightarrow \ker f \xrightarrow{\iota} M \xrightarrow{f} N$$

Then

$$0 \longrightarrow (\ker f)_m \xrightarrow{i_m} M_m \xrightarrow{f_m} N_m$$

is exact at M_m. Since f_m is injective,

$$(\ker f)_m = \operatorname{Im} i_m = \ker f_m = \{0\}.$$

Since each localisation $(\ker f)_m$ of the module $\ker f$ is 0, it follows by previous proposition that

$$\ker f = 0$$

So $f: M \longrightarrow N$ is injective. $\square$

Likewise, it can be shown that surjectivity is also a local property