Kakutani's Theorem

And its application to game theory

Muhammad Haris Rao

18 September 2020

Table of Contents

Convexivity

Definition

A **convex combination** of the set $V = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$ is a linear combination such that the coefficients are all non-negative and sum to 1. The set of all convex combinations of V

$$C = \{\sum_{i=1}^{m} \lambda_i \mathbf{v}_i \mid \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i}^{m} \lambda_i = 1\}$$

is called the **convex hull** of V.

Definition

A set $C \subseteq \mathbb{R}^n$ is said to be **convex** when it contains all of its convex combinations. Equivalently, given any 2 points $c_1, c_2 \in C$, it follows that $\lambda c_1 + (1 - \lambda) c_2 \in C$ for all $\lambda \in [0, 1]$.

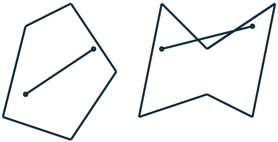

Figure: A convex subset of $\mathbb{R}^2$(left) and a non-convex one (right).

Closure of Convex Hulls

Theorem

The convex hull of a set of vectors $V = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$ is a closed set.

Proof

Let $(\mathbf{x}_n) \to \mathbf{x}_0$ be a sequence in the convex hull of V. Then for each term we have $\mathbf{x}_n = \sum_{i=1}^m \lambda_i^n \mathbf{v}_i$ where each $\lambda_i^n \geq 0$ and $\sum_{i=1}^m \lambda_i^n = 1$ for every $n \in \mathbb{N}$. Then,

$$\mathbf{x}_0 = \lim_{n\to\infty} \mathbf{x}_n = \lim_{n\to\infty} \sum_{i=1}^m \lambda_i^n \mathbf{v}_i = \sum_{i=1}^m \lim_{n\to\infty} (\lambda_i^n) \mathbf{v}_i$$

Since each λ_i^n is in the closed set $[0, 1]$, we know there exists a convergent subsequence for each i. We call the limit of this sequence λ_i^0. Note that λ_i^0 is non-negative. The corresponding subsequence of $(\mathbf{x}_n)$ will still converge to $\mathbf{x}_0$ since it is already convergent. We assume for simplicity that (λ_i^n) itself converges. We've shown that $\mathbf{x}_0$ is a linear combination of V with coefficients λ_i^0. All that is left to do is to show that they sum to 1. Indeed,

$$\sum_{i=1}^m \lambda_i^0 = \sum_{i=1}^m \lim_{n\to\infty} \lambda_i^n = \lim_{n\to\infty} \sum_{i=1}^m \lambda_i^n = \lim_{n\to\infty} 1 = 1$$

Thus, $\mathbf{x}_0$ is a convex combination of V, and the convex hull is closed.

Affine Independence and Simplices

Definition

A the set $V = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$ is said to be **affine independent** when $\sum_{i=1}^{n} \lambda_i \mathbf{v}_i = 0$ and $\sum_{i=1}^{n} \lambda_i = 0$ implies $\lambda_i = 0$ for all i. Equivalently, when the set $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \cdots \mathbf{v}_m - \mathbf{v}_1\}$ is linearly independent.

Definition

An n**-simplex** is the convex hull of an affine independent set of $n+1$ vectors. The **standard** n**-simplex** is the convex hull of the standard basis in $\mathbb{R}^{n+1}$

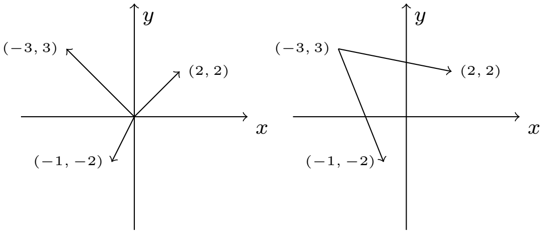

Figure: The vectors $(-3, 3)$, $(-1, -2)$ and $(2, 2)$ being affine independent can be thought of as meaning that the vectors are linearly independent from the point of view of $(-3, 3)$.

Definition

A set valued function $f : A \to B$ is said to be **upper semi-continuous** when given sequences in A $(x_n) \to x$ and $(y_n) \to y$ such that for all $n \in \mathbb{N}$, $x_n \in f(y_n)$, it follows that $x \in f(y)$.

Theorem (Kakutani's Fixed Point Theorem)

Let S be a non-empty, compact and convex subset of $\mathbb{R}^n$, and let $\Phi : S \to \mathcal{P}(S)$ be a set-valued function such that

i. Φ is upper semi-continuous
ii. $\Phi(s)$ is convex for all $s \in S$

Then there exists $s_0 \in S$ such that $s_0 \in \Phi(s_0)$

Review of Game Theory Terms

Definition

A finite, n-person normal-form game is a tuple (N, A, u) where

1. N is a set of n players indexed by i
2. $A = \prod_{i=1}^{n} A_i$ is the set of action profiles, where each A_i is the set of actions available to player i.
3. $u = (u_1, u_2, \cdots u_n)$ is a utility function $u : A \to \mathbb{R}^n$, where each $u_i : A \to \mathbb{R}$ is a real valued utility function for player i

Definition

The mixed extension of the n-person normal-form game (N, A, u) is the tuple (N, S, U) where

1. $S = \prod_{i=1}^{n} S_i$ where S_i is the set of all probability distributions over A_i
2. $U = (U_1, U_2, \cdots, U_n)$ where each U_i is defined as a function $U_i : S \to \mathbb{R}$

$$U_i(s) = \sum_{\mathbf{a} \in A} u_i(\mathbf{a}) \prod_{i=1}^{n} s_i(\mathbf{a}_i)$$

Review of Game Theory Terms Cont.

Definition

The best response function for player i is a function $B_i : S_{-i} \to \mathcal{P}(S_i)$ is the function

$$B_i(s_{-i}) = \{s_i^* \in S_i \mid U_i(s_i^*, s_{-i}) \geq U_i(s_i, s_{-i}) \text{ for all } s_i \in S_i\}$$

We extend the best reponse function and Define $B : S \to \mathcal{P}(S)$

$$B(s) = \prod_{i=1}^{n} B_i(s_{-i})$$

Where $\prod$ denotes the cartesian product. The function B takes in mixed strategies, and returns a the optimal strategies that each player should have played for the highest expected utility. This motivates the following (re)definition:

Definition

A mixed strategy $s \in S$ is said to be a **Nash equilibrium** when $s \in B(s)$.

This is because $s \in B(s)$ means that for each player, $s_i \in B_i(s_{-i})$ and so the strategy profile for each player is already the optimal strategy given the other player's strategies.

It is now more clear how Kakutani's theorem applies to the existence of nash equilibria for finite games. If we can show that B fulfils the conditions where Kakutani's theorem applies, we have proven the existence of nash equilibria.
It has already been argued previously that each S_i, forms a standard $n-1$-simplex when the n actions available to player i are interpreted as the standard basis vectors in $\mathbb{R}^n$ and their probailites as the coeffiecients in a linear combination. So each S_i is convex and closed. To fulfil the conditions of Kakutani's theorem, we need to show that

1. S is compact and convex,
2. B is upper semi-continuous,
3. $B(s)$ is convex for all $s \in S$.

Applications to Game Theory Cont.

Lemma

The cartesian product of finitely many convex sets is convex

Proof

Let $(x_1, x_2, \cdots, x_n), (x_1', x_2', \cdots, x_n') \in X_1 \times X_2 \times \cdots \times X_n$ where each X_i is a convex subset of some euclidean space of arbitrary dimension. Consider the point $\lambda(x_1, \cdots, x_n) + (1-\lambda)(x_1', \cdots, x_n')$ where $\lambda \in [0,1]$. Then see that

$$\lambda(x_1, x_2, \cdots, x_n) + (1-\lambda)(x_1', x_2' \cdots, x_n') = (\lambda x_1 + (1-\lambda)x_1', \cdots, \lambda x_n + (1-\lambda)x_n')$$

Since each X_i is convex, $\lambda x_i + (1-\lambda)x_i' \in X_i$ for each i, and so

$$(\lambda x_1 + (1-\lambda)x_1', \lambda x_2 + (1-\lambda)x_2', \cdots, \lambda x_n + (1-\lambda)x_n') \in X_1 \times X_2 \times \cdots \times X_n$$

Thus, $X_1 \times X_2 \times \cdots \times X_n$ is convex.

Applications to Game Theory Cont.

Lemma

The cartesian product of finitely many closed sets is closed

Proof

Let $X_1, X_2, \cdots, X_k$ be closed sets and let $(x_1^n, x_2^n, \cdots, x_k^n) \to (x_1, x_2, \cdots, x_k)$ be a sequence where $(x_1^n, x_2^n, \cdots, x_k^n) \in X_1 \times X_2, \times \cdots \times X_k$ for all $n \in \mathbb{N}$. Then we have for each $1 \leq i \leq k$, we have the sequence $(x_i^n) \to x_i$ where $x_i^n \in X_i$ for all n. Then since X_i is closed, it must be be true that $x_i \in X_i$. This is true for each i, so it follows that $(x_1, x_2, \cdots, x_k) \in X_1 \times X_2, \times \cdots \times X_k$. Thus, the cartesian product is indeed closed.

The set of mixed strategies for player i, S_i is known to be a closed and convex simplex. It follows fromt the previous 2 lemmas that the set of all mixed strategy profiles $S = S_1 \times S_2 \times \cdots \times S_n$ is also closed and convex.

Applications to Game Theory

Definition

A set valued function $f : A \to B$ is said to be **upper semi-continuous** when given sequences in A $(x_n) \to x$ and $(y_n) \to y$ such that for all $n \in \mathbb{N}$, $x_n \in f(y_n)$, it follows that $x \in f(y)$.

Lemma

The best response function $B(s) : S \to S$ is upper semi-continuous.

Proof Sketch

1. Consider sequences in S, $(r^n) \to r^0$ and $(s^n) \to s^0$ be sequences in S such that $r^n \in B(s^n)$ for all $n \in \mathbb{N}$. Assume for contradtiction that $r^0 \notin B(s^0)$
2. Using the fact that U_i is continuous, show that somewhere along the sequence r^n stopped being the best response to s^n which would be a contradiction.

Proof

Let $(r^n) \to r^0$ and $(s^n) \to s^0$ be sequences in S. Let it also be true that $r^n \in B(s^n)$ for all $n \in \mathbb{N}$. Assume for contradiction that B is not upper semi-continuous, so $r^0 \notin B(s^0)$. Then for some i, we have $r_i^0 \notin B_i(s_{-i}^0)$. Then let $r_i' \in B_i(s_{-i}^0)$, so there exists $\varepsilon > 0$ such that $U_i(r_i', s_{-i}^0) > U_i(r^0, s_{-i}^0) + 3\varepsilon$
Morover, we can make $U_i(r_i', s_{-i}^n)$ arbitrariliy close to $U_i(r_i', s_{-i}^0)$ by bringing s_{-i}^n sufficiently close to s_{-i}^0. So for sufficiently large n, $U_i(r_i', s_{-i}^n) > U_i(r_i', s_{-i}^0) - \varepsilon$
Similarly, $U_i(r^n, s_{-i}^n)$ can be made arbitrarily close to $U_i(r^0, s_{-i}^0)$ with sufficiently large n.
So $U_i(r^0, s_{-i}^0) > U_i(r^n, s_{-i}^n) - \varepsilon$
So putting these 3 inequalities together gives

$$U_i(r_i', s_{-i}^n) > U_i(r_i', s_{-i}^0) - \varepsilon > U_i(r^0, s_{-i}^0) + 2\varepsilon > U_i(r^n, s_{-i}^n) + \varepsilon$$

But we had a premise that $r^n \in B(s^n)$, and thus $r_i^n \in B_i(s_{-i}^n)$ for all n. So this is a contradiction and so we conclude that B is upper semi-continuous as desired.

This fulfils the second requirement

Applications to Game Theory

Lemma

For all $s \in S$, the set $B(s)$ is convex

Proof

Let $s_k \in S_k$ be such that $s_k = \lambda s_{k_1} + (1-\lambda)s_{k_2}$ where $s_{k_1}, s_{k_2} \in S_k$ and $0 \leq \lambda \leq 1$. We will first demonstrate that $U_i(s_k, s_{-k}) = \lambda U_i(s_{k_1}, s_{-k}) + (1-\lambda)U_i(s_{k_1}, s_{-k})$ This is easy to show

$$\begin{aligned} U_i(s_1, \cdots, s_k, \cdots s_n) &= U_i(s_1, \cdots, \lambda s_{k_1} + (1-\lambda)s_{k_2}, \cdots s_n) \\ &= \sum_{\mathbf{a} \in A} u_i(\mathbf{a}) \left(\prod_{j=1}^{k-1} s_j(a_j) \right) \left(\lambda s_{k_1}(a_k) + (1-\lambda)s_{k_2}(a_k) \right) \left(\prod_{j=k+1}^{n} s_j(a_j) \right) \\ &= \sum_{\mathbf{a} \in A} u_i(\mathbf{a}) \left(\prod_{j=1}^{k-1} s_j(a_j) \right) (\lambda s_{k_1}(a_k)) \left(\prod_{j=k+1}^{n} s_j(a_j) \right) \\ &+ \sum_{\mathbf{a} \in A} u_i(\mathbf{a}) \left(\prod_{j=1}^{k-1} s_j(a_j) \right) ((1-\lambda)s_{k_2}(a_k)) \left(\prod_{j=k+1}^{n} s_j(a_j) \right) \end{aligned}$$

Applications to Game Theory

$$
\begin{aligned}
U_i(s_1, \cdots, s_k, \cdots s_n) &= \lambda \sum_{\mathbf{a} \in A} u_i(\mathbf{a}) \left(\prod_{j=1}^{k-1} s_j(a_j) \right) s_{k_1}(a_k) \left(\prod_{j=k+1}^{n} s_j(a_j) \right) \\
&\quad + (1-\lambda) \sum_{\mathbf{a} \in A} u_i(\mathbf{a}) \left(\prod_{j=1}^{k-1} s_j(a_j) \right) s_{k_2}(a_k) \left(\prod_{j=k+1}^{n} s_j(a_j) \right) \\
&= \lambda U_i(s_1, \cdots, s_{k_1}, \cdots, s_n) + (1-\lambda) U_i(s_1, \cdots, s_{k_2}, \cdots, s_n)
\end{aligned}
$$

The rest of the proof is straightforward as well. Let $s_{-i} \in S_{-i}$ and $b_1, b_2 \in B_i(s_{-i})$. Then $U_i(b_1) = U_i(b_2)$ and so

$$
\begin{aligned}
U_i(\lambda b_1 + (1-\lambda) b_2, s_{-i}) &= \lambda U_i(b_1, s_{-i}) + (1-\lambda) U_i(b_2, s_{-i}) \\
&= \lambda U_i(b_1, s_{-i}) + (1-\lambda) U_i(b_1, s_{-i}) \\
&= U_i(b_1, s_{-i})
\end{aligned}
$$

Then $\lambda b_1 + (1-\lambda) b_2 \in B_i(s_{-i})$, so $B_i(s_{-i})$ is convex for all $s_{-i} \in S_{-i}$ each player i. Thus, $B(s)$, the cartesian product of each $B_i(s_{-i})$ is also convex.

All requirements for Kakutani's Theorem have now been demonstrated. We can now prove Nash's theorem for the existence of Nash equilibria for mixed extensions.

Applications to Game Theory Cont.

Theorem (Nash's Theorem)

Every mixed extension of a finite n-person normal-form game has a Nash equilibrium

Proof

Let (N, S, U) be a mixed extension of a finite n-person normal-form game. It has been shown that S is a compact and convex set, and that the function $B : S \to S$ is upper-semi continuous. Moreover, $B(s)$ is convex for every $s \in S$. Then applying Kakutani's fixed point theorem, there exists $s_0 \in S$ such that $s_0 \in B(s_0)$.

Subdivisions and Triangulations

Definition

Let Δ_n be a standard n-simplex. A **triangulation** of Δ_n is a finite collection $S = \{S_1, S_2, S_3, \cdots S_m\}$ such that

1. $S_i \subseteq \Delta_n$ for all $i \in \{1, 2, \cdots, m\}$
2. Each S_i is an n-simplex
3. $S_i \cap S_j$ is empty, or an m-simplex which is a face shared by S_i and S_j whenever $i \neq j$
4. $\cup_{i=1}^{m} S_i = \Delta_n$

Intuitively, a triangulation of an n-simplex is just cutting up the simplex into smaller simplices.

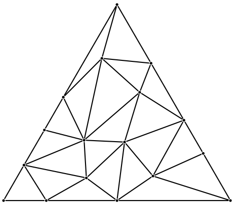

Figure: A triangulation of a standard 2-simplex

Theorem (Brouwer's Fixed Point Theorem)

Let $f : \Delta_n \to \Delta_n$ be a continuous function. Then there exists $x \in \Delta_n$ such that $x = f(x)$.

Theorem (Kakutani's Theorem for n-simplices)

Let $\Phi(x) : \Delta_n \to \mathcal{P}(\Delta_n)$ be an upper semi-continuous function such that $\Phi(x)$ is a convex set for all $x \in \Delta_n$. Then there exists $x_0 \in \Delta_n$ such that $x_0 \in \Phi(x_0)$

Proof Sketch

1. Construct a series of triangulations (T_i) where the mesh goes to 0
2. Define a function φ_i on each triangulation which maps each vertex of T_i to some element of its image under Φ
3. Extend the function linearly over each sub-simplex. Appeal to Brouwer's fixed point theorem and denote the fixed point as $\mathbf{x}^i$.
4. The sequence of $\mathbf{x}^i$ is in a compact set, so denote the limit $\mathbf{x}^0$.
5. Each $\mathbf{x}^i$ is a convex combination of the vertices of the simplex in T_i it is contained in. For each T_i, number the vertices of this simplex with $1, 2, \cdots, n+1$. The vertices numbered 1 form a sequence, and so do those numbered 2, and so on.
6. We also define for each sequence of vertices a sequence of the image of each vertex under φ_i. We have $n+1$ sequences of vertices, and their corresponding images under φ_i. Each has convergent subsequence.
7. As $i \to \infty$, the mesh approaches 0 so each sequence of vertices in fact approaches $\mathbf{x}^0$.
8. The sequence of vertices approaches the limit $\mathbf{x}^0$, and the sequence of the images of the vertices under φ_i approaches a limit. Each term of the latter is contained within the image of each term of the former under Φ. By upper semi-continuity, the limit of the images of the vertices under φ_i is contained in the image of $\mathbf{x}^0$ under Φ.
9. Show that $\mathbf{x}^0$ is in fact a convex combination of the limits of the images of the vertices under φ_i.
10. The result follows from the fact that Φ maps to convex sets

Kakutani's Theorem

Theorem (Kakutani's Theorem for n-simplices)

Let $\Phi(x) : \Delta_n \to \mathcal{P}(\Delta_n)$ be an upper semi-continuous function such that $\Phi(x)$ is a convex set for all $x \in \Delta_n$. Then there exists $x_0 \in \Delta_n$ such that $x_0 \in \Phi(x_0)$

Proof

We again have a sequence of triangulations $(T_i)_{i\in\mathbb{N}}$ on the n-simplex Δ_n such that the mesh approaches 0. Let $V_i = \{\mathbf{v}_1^i, \mathbf{v}_2^i, \mathbf{v}_3^i, \cdots \mathbf{v}_{n_i}^i\}$ be the vertices in V_i. For each triangulation T_i, we will define a function $\varphi_i : \Delta_n \to \Delta_n$ as follows.
For each vertex $\mathbf{v}_j^i \in V_i$, we will define $\varphi_i(\mathbf{v}_j^i)$ as any of the vectors in $\Phi(\mathbf{v}_j^i)$. That is, $\varphi_i(\mathbf{v}_j^i) \in \Phi(\mathbf{v}_j^i)$ for all $\mathbf{v}_j^i \in V_i$. Next we extend φ_i linearly over each simplex in T_i. That is, if $\mathbf{x}$ is in the simplex with vertices $\mathbf{u}_1^i, \mathbf{u}_2^i, \cdots, \mathbf{u}_{n+1}^i$, then

$$\mathbf{x} = \sum_{j=1}^{n} \alpha_j^i \mathbf{u}_j^i$$

Where $\alpha_k^i \geq 0$ for all $k \in \{1, 2, 3, \cdots n+1\}$ and $\sum_{j=1}^{n} \alpha_j^i = 1$. Then $\varphi_i(\mathbf{x})$ is defined as

$$\varphi_i(\mathbf{x}) = \varphi_i\left(\sum_{j=1}^{n+1} \alpha_j^i \mathbf{u}_j^i\right) = \sum_{j=1}^{n+1} \alpha_j^i \varphi_i(\mathbf{u}_j^i)$$

Kakutani's Theorem Cont.

See that each $\varphi_i : \Delta_n \to \Delta_n$ is a continuous function between compact sets. So by Brouwer's fixed point theorem, there is a fixed point for each φ_i which we will denote $\mathbf{x}^i$. Such a point is defined for each T_i, so in fact we have a sequence of points $(\mathbf{x}^i)_{i\in\mathbb{N}}$ which is clearly bounded in the compact set Δ_n. So there is a convergent subsequence. To avoid having to use more complicated indexing, we assume $(\mathbf{x}^i)_{i\in\mathbb{N}}$ is indeed such a subsequence and $\lim_{i\to\infty} \mathbf{x}^i = \mathbf{x}^0$. We claim this is the desired fixed point.

Each $\mathbf{x}^i$ is contained in a simplex of T_i with vertices $\mathbf{w}^i_1, \mathbf{w}^i_2, \cdots, \mathbf{w}^i_{n+1}$. So we also have bounded sequences $(\mathbf{w}^i_k)_{i\in\mathbb{N}}$ for each $k \in \{1, 2, \cdots, n+1\}$, and these also contain convergent subsequences. We again assume they are themselves are convergent for simplicity and for each k, $\lim_{i\to\infty} \mathbf{w}^i_k = \mathbf{w}^0_k$. Moreover, $\mathbf{x}^i$ may be expressed as a convex combination

$$\mathbf{x}^i = \sum_{j=1}^{n+1} \lambda^i_j \mathbf{w}^i_j$$

With the usual restriction on each λ^i_j. The image of each $\mathbf{w}^i_j$ under φ_i will be denoted $\mathbf{y}^i_j$. Because the sequence $(\mathbf{w}^i_k)_{i\in\mathbb{N}}$ converges, the sequence $(\mathbf{y}^i_k)_{i\in\mathbb{N}}$ will also converge since φ_i is continuous. Let this limit be $\mathbf{y}^0_k$. Similarly, let $(\lambda^i_k)_{i\in\mathbb{N}} \to \lambda^0_k$.

Kakutani's Theorem Cont.

Putting all these limits together gives the following

$$\mathbf{x}^0 = \lim_{i\to\infty} \mathbf{x}^i = \lim_{i\to\infty} \varphi_i(\mathbf{x}^i) = \lim_{i\to\infty} \varphi_i\left(\sum_{j=1}^{n+1} \lambda_j^i \mathbf{w}_j^i\right) = \lim_{i\to\infty}\left(\sum_{j=1}^{n+1} \lambda_j^i \varphi_i(\mathbf{w}_j^i)\right)$$

$$= \lim_{i\to\infty}\left(\sum_{j=1}^{n+1} \lambda_j^i \mathbf{y}_j^i\right) = \sum_{j=1}^{n+1} \lim_{i\to\infty} \lambda_j^i \lim_{i\to\infty} \mathbf{y}_j^i = \sum_{j=1}^{n+1} \lambda_j^0 \mathbf{y}_j^0$$

Moreover,

$$\sum_{j=1}^{n+1} \lambda_j^0 = \sum_{j=1}^{n+1} \lim_{i\to\infty} \lambda_j^i = \lim_{i\to\infty} \sum_{j=1}^{n+1} \lambda_j^i = \lim_{i\to\infty} 1 = 1$$

Each λ_k^0 is in the set $[0,1]$ since they are limits of sequences in this closed set. So $\mathbf{x}^0$ is a convex combination of the set $\{\mathbf{y}_1^0, \mathbf{y}_2^0, \cdots, \mathbf{y}_{n+1}^0\}$.

Going back to the sequence $(\mathbf{w}_k^i)_{i\in\mathbb{N}}$, we said that this conveges to $\mathbf{w}_k^0$ for each k. We defined these to be the vertices of the simplex in V_i containing the fixed point $\mathbf{x}^i$. The simpleces in the triangulations all approach 0, so it is not hard to show that each of these sequence in fact converges to the limit of $(\mathbf{x}^i)_{i\in\mathbb{N}}$. That is, $\mathbf{x}^0$.

Kakutani's Theorem Cont.

So we have for each triangulation k the following

$$(\mathbf{w}_k^i)_{i\in\mathbb{N}} \to \mathbf{x}^0$$

$$(\mathbf{y}_k^i)_{i\in\mathbb{N}} \to \mathbf{y}_k^0$$

$$\mathbf{y}_k^i = \varphi_i(\mathbf{w}_k^i) \in \Phi(\mathbf{w}_k^i)$$

From the upper semi-continuity of Φ, it follows that $\mathbf{y}_k^0 \in \Phi(\mathbf{x}^0)$ for each k. But one of the premises is that Φ maps to convex sets. So any convex combination of $\{\mathbf{y}_1^0, \mathbf{y}_2^0, \cdots, \mathbf{y}_{n+1}^0\}$ must also be in $\Phi(\mathbf{x}^0)$. We have already shown that $\mathbf{x}^0$ is such a convex combination, so we conclude that $\mathbf{x}^0 \in \Phi(\mathbf{x}^0)$ as desired.